$$= \sum_{k=1}^{n} f(x_k)\{\alpha(x_k) - \alpha(t_k)\} +$$

$$\sum_{k=1}^{n} f(x_{k-1})\{\alpha(t_k) - \alpha(x_{k-1})\}$$

$$= f(x_0)\{\alpha(t_1) - \alpha(x_0)\} + f(x_1)\{\alpha(x_1) - \alpha(t_1)\}$$

$$+ f(x_1)\{\alpha(t_2) - \alpha(x_2)\} + f(x_2)\{\alpha(x_2) - \alpha(t_2)\}$$

$$+ \dots + f(x_{n-1})\{\alpha(t_n) - \alpha(x_{n-1})\} + f(x_n)\{\alpha(x_n) - \alpha(t_n)\}$$

$$A' - S(P, \alpha, f) = S(P', f, \alpha) \longrightarrow (4)$$

$\therefore$ where P' is a partition of $[a,b]$ obtained by considering x_k and t_k together

Let $A = f(b)\alpha(b) - f(a)\alpha(a) - \int_a^b f(x)\, d[\alpha(x)] \rightarrow (5)$

Let $P \supseteq P_\epsilon$ we know that $P' \supseteq P$

This implies $P' \supseteq P \supseteq P_\epsilon$

$$\Rightarrow P' \supseteq P_\epsilon$$

For such P, equation (1) holds good.

Consider,

$$|S(P,\alpha,f) - A| = |S(P,\alpha,f) - A' + \int_a^b f\, d\alpha|$$

$$= |A' - S(P', f, \alpha) - A' + \int_a^b f\, d\alpha|$$

$$= |-S(P', f, \alpha) + \int_a^b f\, d\alpha|$$

$$= |S(P', f, \alpha) - \int_a^b f\, d\alpha|$$

$$< \epsilon$$

$\Rightarrow |S(P, \alpha, f) - A| < \epsilon$

$\Rightarrow \alpha \in R(f)$ on $[a,b]$ and $A = \int_a^b \alpha\, df$

$\Rightarrow \alpha \in R(f)$ on $[a,b]$ and $A' = \int_a^b f\, d\alpha = \int_a^b \alpha\, df$

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(ii) $\alpha \in R(f)$ on $[a,b]$ and $A' = \int_a^b f\,d\alpha + \int_a^b \alpha\,df$

where $A' = f(b)\alpha(b) - f(a)\alpha(a)$

$$\therefore \int_a^b f(x)\,d[\alpha(x)] + \int_a^b \alpha(x)\,d(f(x)) = f(b)\alpha(b) - f(a)\alpha(a)$$

Hence proved

Theorem: 6

Q. State and prove change of variable theorem in Riemann - Stieltjes Integral.

Statement:

Let $f \in R(\alpha)$ on $[a,b]$ and let g be a strictly monotonic continuous function defined on an Interval S having end points c and d

Assume that $a = g(c)$, $b = g(d)$

Let h and β be the composite functions defined as follows

$$h(x) = f[g(x)] \text{ and } \beta(x) = \alpha[g(x)] \text{ if } x \in S$$

then $h \in R(\beta)$ on S and we have,

$$\int_a^b f\,d\alpha = \int_c^d h\,d\beta$$

that is $$\int_{g(c)}^{g(d)} f(t)\,d\alpha(t) = \int_c^d f[g(x)]\,d\{\alpha[g(x)]\}$$

Proof:

Given, $f \in R(\alpha)$ on $[a,b]$

g is a strictly monotonic continuous function on $S = [c,d]$

And $g(c) = a$, $g(d) = b$

$\therefore g : [c,d] > [a,b]$

For definiteness, assume that g is strictly increasing on $S = [c,d]$ (this implies $c < d$)

Also g is continuous function defined on the compact set $S = [c,d]$

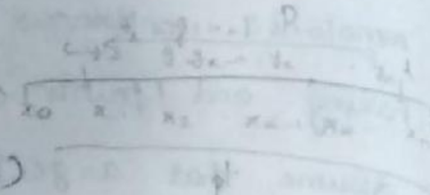
This implies, g^{-1} is continuous and also g^{-1} is strictly increasing on $[c,d]$

$\therefore$ For every Partition $P = \{y_0, y_1, \dots y_n\}$ of $[c,d]$ there corresponds one and only partition.

$P' = \{x_0, x_1, \dots x_n\}$ of $[a,b]$ with

$x_k = g(y_k)$; $k = 0,1,2,\dots n$

Then we can write,



$P' = g(P)$ and $P = g^{-1}(P')$

Furthermore, Refinement of P produces a corresponding refinement of P' conversely since $f \in R(\alpha)$ on $[a,b]$

For any given $\epsilon > 0$ $\exists$ a partition P_ϵ' of $[a,b]$

Such that $P' \supseteq P_\epsilon'$ we have,

$$\left| S(P', f, \alpha) - \int_a^b f \, d\alpha \right| < \epsilon \longrightarrow (1)$$

Let $P_\epsilon = g^{-1}(P_\epsilon')$

ie) (let P_ϵ of $[c,d]$ be the corresponding partition of P_ϵ' of $[a,b]$)

Let $A = \int_a^b f \, d\alpha$

Let $P = \{y_0, y_1, \dots y_n\}$ be a partition of $[c,d]$ finer than P_ϵ

ie) $P \supseteq P_\epsilon$

Consider the Riemann - Stieltjes sum

$$S(P, h, \beta) = \sum_{k=1}^{n} h(u_k)\, \Delta\beta_k$$

where $u_k \in [y_{k-1}, y_k]$ and

$$\Delta\beta_k = \beta(y_k) - \beta(y_{k-1})$$

If we put $t_k = g(u_k)$ and $x_k = g(y_k)$

then $t_k \in [x_{k-1}, x_k]$.

Then the partition $P' = \{x_0, x_1, \dots x_n\}$ of $[a,b]$ is finer than P'_ϵ

$$P' \supseteq P'_\epsilon$$

equation ① is valid for such P'.

then for all $P' \supseteq P'_\epsilon$

consider, $|S(P, h, \beta) - A|$

$$|S(P, h, \beta) - \int_a^b f\,d\alpha|$$

$$= \left|\sum_{k=1}^{n} h(u_k)\,\Delta\beta_k - \int_a^b f\,d\alpha\right|$$

$$= \left|\sum_{k=1}^{n} h(u_k)\{\beta(y_k) - \beta(y_{k-1})\} - \int_a^b f\,d\alpha\right| \quad h = f(g(x)),\ \beta = \alpha(g(x))$$

$$= \left|\sum_{k=1}^{n} f(g(u_k))\{\alpha(g(y_k)) - \alpha(g(y_{k-1}))\} - \int_a^b f\,d\alpha\right|$$

$$= \left|\sum_{k=1}^{n} f(t_k)\{\alpha(x_k) - \alpha(x_{k-1})\} - \int_a^b f\,d\alpha\right| \quad g(u_k) = t_k,\ x_k = g(y_k)$$

$$= \left|\sum_{k=1}^{n} f(t_k)\,\Delta\alpha_k - \int_a^b f\,d\alpha\right|$$

$$= \left|S(P', f, \alpha) - \int_a^b f\,d\alpha\right|$$

$< \epsilon$ By (1)

$\Rightarrow h \in R(\beta)$ on $S = [c, d]$ and $A = \int_c^d h \, d\beta$

i.e., $h \in R(\beta)$ on S and $\int_a^b f \, d\alpha = \int_c^d h \, d\beta$

(ie) $$\int_{g(c)}^{g(d)} f(t) \, d[\alpha(t)] = \int_c^d f[g(x)] \cdot d[\alpha(g(x))]$$

Hence the proof.

Theorem: 6

Reduction to Riemann - Integral :-

Assume $f \in R(\alpha)$ on $[a, b]$ and assume that α has a continuous derivative α' on $[a, b]$. Then the Riemann - integral $\int_a^b f(x) \, \alpha'(x) \, dx$ exists and we have

$$\int_a^b f(x) \, d[\alpha(x)] = \int_a^b f(x) \, \alpha'(x) \, dx$$

Proof:

Given: $f \in R(\alpha)$ on $[a, b]$ and α has a continuous derivative α' on $[a, b]$.

To prove that:

$$\int_a^b f(x) \, \alpha'(x) \, dx \text{ exists}$$

Let $g(x) = f(x) \, \alpha'(x)$

Tp:- $|S(P, g) - A| < \epsilon$ T.P: $|S(P,g) - A| = |S(P,g) - S(P,f,\alpha) + \ldots$

Consider the Riemann sum for a partition.

$$P = \{x_0, x_1, x_2, \ldots, x_n\} \text{ of } [a, b]$$

$$S(P, g) = \sum_{k=1}^{n} g(t_k) \, \Delta x_k \quad \text{where } t_k \in [x_{k-1}, x_k]$$

$$= \sum_{k=1}^{n} f(t_k) \, \alpha'(t_k) \, \Delta x_k \longrightarrow (1)$$

where, $\Delta x_k = x_k - x_{k-1}$

For the same partition P and for the same choice of t_k, consider

The Riemann - stieltjes sum,

$$S(P, f, \alpha) = \sum_{k=1}^{n} f(t_k)\, \Delta\alpha_k \longrightarrow (2)$$

Since α is continuous on $[a, b]$ and α' exists on $[a, b]$ we have,

α is continuous on $[x_{k-1}, x_k]$ and α' exists on $[x_{k-1}, x_k]$

Applying Lagrange Mean Value theorem for the function α' in $[x_{k-1}, x_k]$ we have

$\exists\ V_k \in [x_{k-1}, x_k]$ Such that $\qquad f'(c) = \dfrac{f(b) - f(a)}{b-a}$

mean value thm $\quad \alpha'(V_k) = \dfrac{\alpha(x_k) - \alpha(x_{k-1})}{x_k - x_{k-1}}$

$$\Rightarrow \alpha'(V_k)\,[x_k - x_{k-1}] = \alpha(x_k) - \alpha(x_{k-1})$$

$$\Rightarrow [x_k - x_{k-1}]\,\alpha'(V_k) = \Delta\alpha_k \longrightarrow (3)$$

Using (3) in (2) we have

$$S(P, f, \alpha) = \sum_{k=1}^{n} f(t_k)\,\alpha'(V_k)\,(x_k - x_{k-1})$$

$$\Rightarrow S(P, f, \alpha) = \sum_{k=1}^{n} f(t_k)\,\alpha'(V_k)\,\Delta x_k \longrightarrow (4)$$

(4) - (1) gives,

$$\Rightarrow S(P, f, \alpha) - S(P, g) = \sum_{k=1}^{n} f(t_k)\left\{\alpha'(V_k) - \alpha'(t_k)\right\}\Delta x_k$$

$\longrightarrow (5)$

Since f is bounded on $[a, b]$, there exists $M > 0$, Such that

$$|f(x)| < M \quad \forall\ x \in [a, b] \longrightarrow (6)$$

Since α' is continuous on a compact set $[a,b]$

α' is uniformly continuous on $[a,b]$

$\therefore$ For any given $\epsilon > 0$, $\exists$ $\delta > 0$ (δ depends only on ϵ) such that,

$$0 \le |x-y| < \delta \Rightarrow |\alpha'(x) - \alpha'(y)| < \frac{\epsilon}{2M(b-a)} \longrightarrow (7)$$

Let P_ϵ' be a partition of $[a,b]$ with $\|P_\epsilon'\| < \delta$.

Then for $P \supseteq P_\epsilon'$ we have

$$\Longrightarrow |v_k - t_k| < \delta$$

[$\because P \supseteq P_\epsilon' \Rightarrow \|P\| \le \|P_\epsilon'\| < \delta$

$\Rightarrow |x_k - x_{k-1}| < \delta \ \forall k$

$\Rightarrow |v_k - t_k| < \delta$

$\|P\| < \delta \Rightarrow$ length of largest subinterval of P is $< \delta$ then length of every subinterval is $< \delta$]

$\therefore (7) \Rightarrow \forall P \supseteq P_\epsilon'$

$$|\alpha'(v_k) - \alpha'(t_k)| < \frac{\epsilon}{2M(b-a)} \longrightarrow (8)$$

$(5) \Rightarrow \forall P \supseteq P_\epsilon'$

$$|S(P,f,\alpha) - S(P,g)| = \left| \sum_{k=1}^{n} f(t_k) \{\alpha'(v_k) - \alpha'(t_k)\} \right| \Delta x_k$$

$$\le \sum_{k=1}^{n} |f(t_k)| \, |\alpha'(v_k) - \alpha'(t_k)| \, \Delta x_k$$

$$\le \sum_{k=1}^{n} M \frac{\epsilon}{2M(b-a)} \Delta x_k$$

$$< M \frac{\epsilon}{2M(b-a)} \sum_{k=1}^{n} \Delta x_k$$

$$< \frac{\epsilon}{2(b-a)} (b-a)$$

$$= \frac{\epsilon}{2}$$

$\left\{ \sum_{k=1}^{n} \Delta x_k = \sum_{k=1}^{n} (x_k - x_{k-1}) = x_n - x_0 = b-a \right\}$

$\therefore \ |S(P,f,\alpha) - S(P,g)| < \frac{\epsilon}{2} \longrightarrow (9)$

Since $f \in R(\alpha)$ on $[a,b]$

for any given $\epsilon > 0$, $\exists$ a partition P_ϵ'' of $[a,b]$

Such that, $\forall \ P \supseteq P_\epsilon''$

$$\left|S(P,f,\alpha) - \int_a^b f \, d\alpha\right| < \frac{\epsilon}{2} \longrightarrow (10)$$

Let $P_\epsilon = P_\epsilon' \cup P_\epsilon''$ then

$P \supseteq P_\epsilon \Rightarrow P \supseteq P_\epsilon'$ and $P \supseteq P_\epsilon''$

$\therefore$ for such P both (9) & (10) hold

Let $A = \int_a^b f \, d\alpha$

then for $P \supseteq P$,

$$|S(P,g) - A| = \left|S(P,g) - S(P,f,\alpha) + S(P,f,\alpha) - \int_a^b f \, d\alpha\right|$$

$$\leq |S(P,g) - S(P,f,\alpha)| + \left|S(P,f,\alpha) - \int_a^b f \, d\alpha\right|$$

$$< \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon$$

$\Rightarrow \ |S(P,g) - A| < \epsilon$

$\therefore \ g \in R$ on $[a,b]$ and $A = \int_a^b g(x) \, dx$

$\Rightarrow \ f(x) \, \alpha'(x) \in R$ and $\int_a^b f(x) \, d(\alpha(x)) = \int_a^b f(x) \, \alpha'(x) \, dx$

i.e) $\int_a^b f(x) \, \alpha'(x) \, dx$ exists and

$$\int_a^b f(x) \, d[\alpha(x)] = \int_a^b f(x) \, \alpha'(x) \, dx$$

Hence the proof.

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7.8 Step function as Integrators

Note:

If α is a constant throughout $[a,b]$, then integral $\int_a^b f\,d\alpha$ exists and has the value 0 (zero), since each sum $S(P,f,\alpha)=0$.

However, if α is constant except for a jump discontinuity at one point, the integral $\int_a^b f\,d\alpha$ need not exist and if it does exist, its value need not be zero.

[i.e. we say the function is continuous ⇒ R-P-L R-S-P exists]

Theorem: 7

* Given $a<c<b$. Define α on $[a,b]$ as follows, the values $\alpha(a)$, $\alpha(b)$, $\alpha(c)$ are arbitrary,

$\alpha(x)=\alpha(a)$ if $a\le x<c$ and

$\alpha(x)=\alpha(b)$ if $c<x\le b$

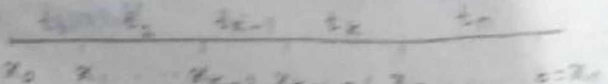
Let f be defined on $[a,b]$ in such a way that atleast one of the functions f or α is continuous from the left at c and atleast one is continuous from the right at c then $f\in R(\alpha)$ on $[a,b]$ and we have,

$$\int_a^b f\,d\alpha = f(c)\left[\alpha(c+)-\alpha(c-)\right]$$

Proof: Let P be a partition of $[a,b]$,

$$P=\{a=x_0, x_1, x_2, \ldots, x_{k-2}, c, x_k, \ldots x_n=b\}$$

Such that $c\in P$



$= \sum_{k=1}^{n} f(t_k)\,\Delta\alpha_k$

Then,

$$S(P,f,\alpha)=f(t_1)\left[\alpha(x_1)-\alpha(x_0)\right]+f(t_2)\left[\alpha(x_2)-\alpha(x_1)\right]$$
$$+\cdots+f(t_{k-1})\left[\alpha(c)-\alpha(x_{k-2})\right]+$$
$$f(t_k)\left[\alpha(x_k)-\alpha(c)\right]+\cdots+f(t_n)\left[\alpha(x_n)-\alpha(x_{n-1})\right]$$

$$\Rightarrow S(P, f, \alpha) = f(t_1)[\alpha(a) - \alpha(a)] + f(t_2)[\alpha(a) - \alpha(a)] + \cdots +$$
$$f(t_{k-1})[\alpha(c) - \alpha(x_{k-2})] + f(t_k)[\alpha(x_k) - \alpha(c)] + \cdots +$$
$$f(t_n)[\alpha(b) - \alpha(b)]$$

$$\Rightarrow S(P, f, \alpha) = 0 + 0 + \cdots + f(t_{k-1})[\alpha(c) - \alpha(c-)]$$
$$+ f(t_k)[\alpha(c+) - \alpha(c)] + \cdots + 0$$
$$= f(t_{k-1})[\alpha(c) - \alpha(c-)] + f(t_k)[\alpha(c+) - \alpha(c)]$$

where $t_{k-1} \leq c \leq t_k$

Let $A = f(c)[\alpha(c+) - \alpha(c-)]$

then,

$$\Delta = S(P, f, \alpha) - A$$
$$\Delta = f(t_{k-1})[\alpha(c) - \alpha(c-)] + f(t_k)[\alpha(c+) - \alpha(c)]$$
$$- f(c)[\alpha(c+) - \alpha(c-)]$$

$$\Delta = f(t_{k-1})[\alpha(c) - \alpha(c-)] + f(t_k)[\alpha(c+) - \alpha(c)]$$

Add & Sub $f(c)\alpha(c)$

$$- f(c)[\alpha(c+) - \alpha(c-)] + f(c)\alpha(c) - f(c)\alpha(c)$$

$$\Delta = f(t_{k-1})[\alpha(c) - \alpha(c-)] + f(t_k)[\alpha(c+) - \alpha(c)]$$
$$- f(c)[\alpha(c) - \alpha(c-)] - f(c)[\alpha(c+) - \alpha(c)]$$

$$\Delta = [f(t_{k-1}) - f(c)][\alpha(c) - \alpha(c-)]$$
$$+ [f(t_k) - f(c)][\alpha(c+) - \alpha(c)] \longrightarrow (1)$$

Hence,

$$|\Delta| \leq |f(t_{k-1}) - f(c)|\,|\alpha(c) - \alpha(c-)|$$
$$+ |f(t_k) - f(c)|\,|\alpha(c+) - \alpha(c)| \longrightarrow (2)$$

Case: (i)

If f is continuous at c,

then for any given $\epsilon > 0$, $\exists\ \delta > 0$ such that,

$\|P\| < \delta \Rightarrow |f(t_{k-1}) - f(c)| < \epsilon$ and